JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 38, No. 3, May–June 2001

Direct High-Speed Interception: Analytic Solutions, Qualitative Analysis, and Applications

Yuri Ulybyshev*
Rocket-Space Corporation "Energia," 141070, Korolev, Russia

Trajectories for high-speed interception in a thin spherical shell of an inverse-square central gravity field are considered. The solutions are obtained for the problem of exoatmospheric flight, open time intercept of a nonmaneuvering target in an orbit during free-flight phase. The fixed-fuel interceptor is assumed to have the capability of generating fixed magnitude thrust or a specific impulsive velocity change. A closed-form solution for the control law is derived. The thrust direction is a linear function of the relative states between the interceptor and target and a nonlinear function of the transfer time. This transfer time is obtained as an analytic solution to a quadratic equation. First- and second-order methods are developed. A qualitative analysis and a descriptive geometric interpretation of a space interception are considered. The results show that the optimal guidance law for the boost phase of an interceptor is the well-known constant bearing course relative to the target. Descriptive existence conditions of trajectories for a fixed-fuel interceptor are derived and a computation method to determine the reachable domain, that is, the boundary set of initial positions of the interceptor and target, is developed. Numerical examples of an asteroid interception and satellite interception are presented.

Nomenclature

a	=	initial thrust acceleration; Eq. (1)
a_i	=	polynomial approximation of δr for free-flight phase; Eq. (B3)
b_i	=	polynomial approximation of δr for boost phase; Eq. (C2)
C_1, C_2	=	constants; Eq. (42)
D	=	discriminant
$\boldsymbol{e}$	=	unit vector of thrust direction
H	=	Hamiltonian function
$\dot{m}$	=	specific fuel usage rate
$\boldsymbol{Q}_v, \boldsymbol{Q}_\rho$	=	perturbed terms; Eqs. (30) and (31)
r	=	radius
$\boldsymbol{r}$	=	radius vector
r_m	=	mean radius of thin spherical shell
S_v, S_ρ	=	perturbed terms; Eqs. (30) and (31)
t	=	time
t'_f	=	minimal range time in an inverse-square gravity field
$\boldsymbol{V}$	=	velocity vector, km/s
W_ρ, W_v	=	scaled distance and velocity changes for homogeneous central gravity field; Eq. (A7)
$\boldsymbol{0}$	=	zero vector
$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}$	=	constant vectors
Δi	=	inclination angle interceptor orbital plane; see Fig. 9
Δt_f	=	$t_f - t_k$
ΔV	=	specific impulsive velocity change, km/s
δr	=	$r - r_m$
δt_f	=	relative displacement of minimal range time, $(t_f - t'_f)/t'_f$
$\delta\rho_{\min}$	=	$\rho_{\min}/\rho_0$
θ	=	angle between vectors $\boldsymbol{V}$ and $\boldsymbol{\rho}$
$\boldsymbol{\lambda}$	=	adjoint vector; Eq. (11)
μ	=	gravitational constant of central body
ρ	=	relative distance between interceptor and target
$\boldsymbol{\rho}$	=	relative position vector
$\rho_{\min}$	=	minimal range in an inverse-square gravity field
φ	=	interceptor or target position angle; see Fig. 9
ω	=	mean motion of circular orbit with radius of r_m; Eq. (5)

Subscripts

F	=	final
G	=	gravity term
I	=	interceptor
K	=	end of boost phase
T	=	target
V	=	velocity
0	=	initial
ρ	=	range

Superscript

T	=	transposition

Introduction

A TOPIC of classical interest in astrodynamics is the interception problem, that is, the determination of a collision trajectory between two vehicles (an interceptor and a target) in space. This problem has been extensively analyzed in the published literature.[1–15] A survey of the earliest works is given by Gobetz and Doll.[1] Recent works can be found in the Refs. 7–14. For ease of discussion these numerous possible methods for solving the interception problem may be categorized in accordance with the gravitational model adopted in each: 1) nongravity model,[2,4] 2) constant gravity field,[3,13,15] 3) homogeneous central gravity field,[3] 4) linearized Clohessy–Wiltshire[16] gravity field,[3,9] and 5) well-known inverse-square central gravity field.[5,7,8,10] For each method there is an application area in terms of the transfer orbit type and transfer angle. Figure 1 is a schematic distribution of the application areas for the cited gravity models.

In the present paper, we consider the high-speed, fixed-fuel interception problem in a thin spherical shell of an inverse-square central gravity field. In this case, a homogeneous central gravity field model[3,17–19] can be used. This field is a simpler model in which gravitational acceleration vector is linear with radius vector.

Previous investigators have obtained solutions that are based on an impulsive thrust approximation. In the case of a direct-ascent interception, the boost-phase time is comparable with the transfer time, and the thrust impulsive approximation is not always applicable. As a rule, the solution to the interception problem for an inverse-square gravity field, for example, in the form of an iterative

Received 24 February 2000; revision received 31 August 2000; accepted for publication 8 January 2001.

*Head, Space Ballistics Department; yuri.ulybyshev@rsce.ru. Member AIAA.

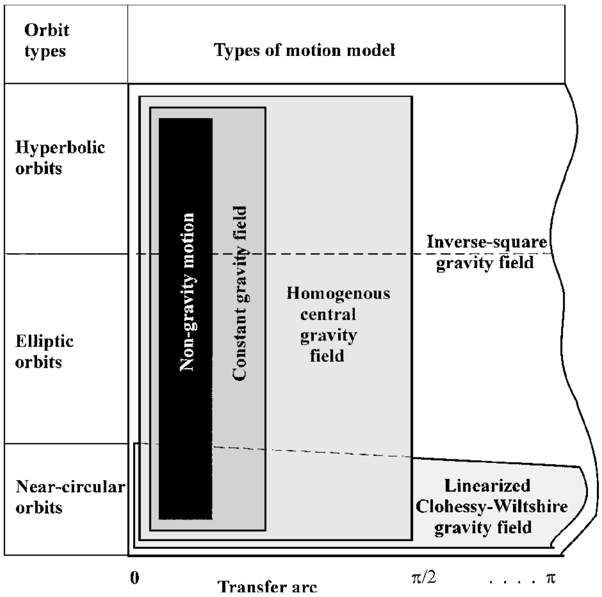

Fig. 1 Schematic diagram for application of gravity field models.

solution of Lambert's problem (see Refs. 6 and 14), is still somewhat complicated. In the case of a homogeneous central gravity field model, the time-optimal solution can be found in closed form for fuel-fixed boost-glide intercept. For these reasons, a simpler, approximate model is useful. In this paper, we derive, analyze, and apply a homogeneous central gravity field model that approximately describes the motion of an interceptor to a target. There are a number applications of this theory, including close-range asteroid interception, space interception, etc.

The complexities of performing an intercept in space are such that an analytical qualitative study is difficult to accomplish. Qualitative analyses and geometric interpretations were presented for the inverse-square central gravity field.[15,20,21] Cochran and Haynes[10] derived a guidance law for exoatmospheric intercept with an impulsive change constraint that is reasonable for solid rocket boosted interceptors. The problem of determining attainable regions for a single-impulse transfer from a given initial Keplerian elliptical orbit has been considered by Kirpichnikov.[20] Vinh et al.[12] introduced the concept of the reachable domain, which is the set of points attainable at a given time by a free-flying interceptor using a fuel potential represented by a total characteristic velocity. They also considered possible applications of this concept. The well-known inverse-square central gravity field is a model with good accuracy for solution of the interception problem. However, only limited explicit solutions and descriptive geometric interpretations have been obtained for this model. Here we present a qualitative analysis and geometric interpretation of the interception problem during the free-flight phase of an interceptor and find a method for explicit computation of reachable domains.

Cochran and Haynes[10] designed a similar technique for energy-limited impulsive interception for specified final position in an inverse-square gravity field based on an iterative procedure including the solution of a quartic equation. An analysis and solutions of optimal intercept trajectories in a homogeneous central gravity field with controlled thrust acceleration are presented in Ref. 3.

The major differences in the present problem and the preceding ones are 1) equations of relative motion between the interceptor and target are considered; 2) the final position, that is, intercept point, is not specified; 3) the transfer time is free; 4) a general solution for a variable mass interceptor with fixed thrust magnitude is derived; and 5) explicit existence conditions, that is, the energetic attainability of the target, are presented.

Analytic Solutions

Equations of Relative Motion

The equations of motion of the interceptor and nonmaneuvering target in an Earth-central inertial frame can be written as

$$\dot{\boldsymbol{V}}_I = -\left(\mu \boldsymbol{r}_I / r_I^3\right) + a\boldsymbol{e}(t)/(1 - \dot{m}t) \tag{1}$$

$$\dot{\boldsymbol{r}}_I = \boldsymbol{V}_I \tag{2}$$

$$\dot{\boldsymbol{V}}_T = -\left(\mu \boldsymbol{r}_T / r_T^3\right) \tag{3}$$

$$\dot{\boldsymbol{r}}_T = \boldsymbol{V}_T \tag{4}$$

where the control $\boldsymbol{e}(t)$ is a unit vector. We assume that the transfer occurs in a thin spherical shell of an inverse-square central gravity field. By this means, the function $\mu/r^3(t)$ may be expanded in a series about a reference value $\omega^2 = \mu/r_m^3$:

$$\mu/r^3 \approx \omega^2(1 - 3\delta r/r_m) \tag{5}$$

The equations of relative motion in terms of relative position $\boldsymbol{\rho} = \boldsymbol{r}_T - \boldsymbol{r}_I$ and relative velocity $\boldsymbol{V} = \boldsymbol{V}_T - \boldsymbol{V}_I$ then are as follows:

$$\dot{\boldsymbol{V}} = -\omega^2\boldsymbol{\rho} - a\boldsymbol{e}(t)/(1 - \dot{m}t) + \left(3\omega^2/r_m\right)(\delta r_T \boldsymbol{r}_T - \delta r_I \boldsymbol{r}_I) \tag{6}$$

$$\dot{\boldsymbol{\rho}} = \boldsymbol{V} \tag{7}$$

First-Order Methods for Interception During Boost-Phase Interception

The following equations shall be used to model the first-order dynamics:

$$\dot{\boldsymbol{V}} = -\omega^2\boldsymbol{\rho} - a\boldsymbol{e}(t)/(1 - \dot{m}t) \tag{8a}$$

$$\dot{\boldsymbol{\rho}} = \boldsymbol{V} \tag{8b}$$

These equations have the form of a harmonic oscillator without damping, but with an applied force (thrust). It is well known that the solution for such harmonic oscillator can be written as

$$\boldsymbol{\rho}(t) = (\boldsymbol{V}_0/\omega)\sin\omega t + \boldsymbol{\rho}_0 \cos\omega t \tag{9a}$$

$$\boldsymbol{V}(t) = \boldsymbol{V}_0 \cos\omega t - \boldsymbol{\rho}_0 \omega \sin\omega t \tag{9b}$$

An optimal control formulation of the problem can be stated as follows. Find the control law $\boldsymbol{e}(t)$ that minimize the flight time t_f to go from prescribed initial conditions $\boldsymbol{\rho}_0$ and $\boldsymbol{V}_0$ to the final conditions

$$\boldsymbol{\rho}(t_f) \equiv 0, \qquad \boldsymbol{V}(t_f) \text{ is free} \tag{10}$$

The variational Hamiltonian[2] is

$$H = \boldsymbol{\lambda}_V^T[\omega^2\boldsymbol{\rho} - a\boldsymbol{e}(t)/(1 - \dot{m}t)] + \boldsymbol{\lambda}_\rho^T \boldsymbol{V} \tag{11}$$

The optimal control law may be determined from the Pontryagin's minimum principle.[2] It is given by

$$\boldsymbol{e}(t) \| \boldsymbol{\lambda}_V(t) \tag{12}$$

The Euler–Lagrange equations are

$$\dot{\boldsymbol{\lambda}}_V = -\frac{\partial H}{\partial \boldsymbol{V}} = -\boldsymbol{\lambda}_\rho \tag{13}$$

$$\dot{\boldsymbol{\lambda}}_\rho = -\frac{\partial H}{\partial \boldsymbol{\rho}} = \omega^2 \boldsymbol{\lambda}_V \tag{14}$$

These equations are easily integrated to yield for $\boldsymbol{\lambda}_v$:

$$\boldsymbol{\lambda}_V(t) = \boldsymbol{\gamma}\sin\omega t + \boldsymbol{\nu}\cos\omega t \tag{15}$$

For the interception trajectories, the final value of relative velocity $\boldsymbol{V}(t_f)$ is unspecified. Because the terminal constraints do not depend on the final velocity, satisfaction of the necessary conditions requires a zero boundary condition for the adjoint vector associated with the velocity, which is equivalent to a zero value,

$$\boldsymbol{\lambda}_V(t_f) = \boldsymbol{\gamma}\sin\omega t_f + \boldsymbol{\nu}\cos\omega t_f \equiv \boldsymbol{0} \tag{16}$$

It follows that the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\nu}$ are collinear vectors. Thus, $\boldsymbol{\lambda}_v$ is a constant direction vector. Therefore, the optimal control law must be given by a constant unit vector $\boldsymbol{e}$. Note that an optimal control problem with constant thrust magnitude of a variable mass interceptor is considered. To our knowledge, the first solution to a similar optimal interception problem with continuous finite thrust acceleration is given by Kouzmak and Braude.[18] The analytic solution of Eqs. (8) for constant thrust direction is presented in Appendix A. Substitution from Eq. (10) into Eq. (A5) yields

$$\boldsymbol{V}_0 \sin\omega t_f + \boldsymbol{\rho}_0\omega\cos\omega t_f + W_\rho(t_f)\boldsymbol{e} = \boldsymbol{0} \tag{17}$$

where W_ρ is a scaled distance of the interceptor depending on the thrust force. Because $|\boldsymbol{e}| \equiv 1$, we have

$$V_0^2 \sin^2\omega t_f + 2\boldsymbol{V}_0^T\boldsymbol{\rho}_0\omega\sin\omega t_f\cos\omega t_f + \rho_0^2\omega^2\cos^2\omega t_f = W_\rho^2(t_f) \tag{18}$$

which is a transcendental equation for t_f. The expression for the control law can be written as

$$\boldsymbol{e} = -\frac{\boldsymbol{V}_0\sin\omega t_f + \boldsymbol{\rho}_0\omega\cos\omega t_f}{W_\rho(t_f)} \tag{19}$$

First-Order Method for Interception During Free-Flight Phase

Consider the minimum-time interception during the free-flight phase of the fixed-fuel interceptor. From Eq. (10) may be found the constraint relation for state vector at the end of the boost phase:

$$\boldsymbol{\rho}(t_f) = (\boldsymbol{V}_k/\omega)\sin\omega\Delta t_f + \boldsymbol{\rho}_k\cos\omega\Delta t_f = 0 \tag{20}$$

where $\Delta t_f = t_f - t_k$ and

$$\boldsymbol{\rho}_k = (\boldsymbol{V}_0/\omega)\sin\omega t_k + \boldsymbol{\rho}_0\cos\omega t_k + (W_{\rho k}/\omega)\boldsymbol{e} \tag{21}$$

$$\boldsymbol{V}_k = \boldsymbol{V}_0\cos\omega t_k - \boldsymbol{\rho}_0\omega\sin\omega t_k + W_{vk}\boldsymbol{e} \tag{22}$$

From $\boldsymbol{\lambda}_v$ being a constant direction vector, it follows that $\boldsymbol{e}$ is also a constant unit vector. We may define

$$\boldsymbol{\rho}_{kg} = (\boldsymbol{V}_0/\omega)\sin\omega t_k + \boldsymbol{\rho}_0\cos\omega t_k \tag{23}$$

$$\boldsymbol{V}_{kg} = \boldsymbol{V}_0\cos\omega t_k - \boldsymbol{\rho}_0\omega\sin\omega t_k \tag{24}$$

and express Eqs. (21) and (22) more compactly as functions of $\boldsymbol{e}$. Therefore,

$$\boldsymbol{\rho}_k = \boldsymbol{\rho}_{kg} + (W_{\rho k}/\omega)\boldsymbol{e} \tag{25}$$

$$\boldsymbol{V}_k = \boldsymbol{V}_{kg} + W_{vk}\boldsymbol{e} \tag{26}$$

Substituting these results for $\boldsymbol{\rho}_k$ and $\boldsymbol{V}_k$ in Eq. (20) yields

$$\boldsymbol{V}_{kg}\sin\omega\Delta t_f + \boldsymbol{\rho}_{kg}\omega\cos\omega\Delta t_f + \boldsymbol{e}[W_{vk}\sin\omega\Delta t_f + W_{\rho k}\cos\omega\Delta t_f] = \boldsymbol{0} \tag{27}$$

Similar to the preceding section, because $|\boldsymbol{e}| \equiv 1$, the last equation simplifies to

$$\left(V_{kg}^2 - W_v^2\right)\tan^2\omega\Delta t_f + 2\left(\boldsymbol{V}_{kg}^T\boldsymbol{\rho}_{kg}\omega - W_{vk}W_{\rho k}\right)\tan\omega\Delta t_f + \left(\rho_{kg}^2\omega^2 - W_{\rho k}^2\right) = 0 \tag{28}$$

which is a quadratic equation for $\tan\omega\Delta t_f$.

For the impulsive interception (i.e., $W_{vk} \equiv \Delta V$, $W_{\rho k} \equiv 0$, $\boldsymbol{V}_{kg} \equiv \boldsymbol{V}_0$, $\boldsymbol{\rho}_{kg} \equiv \boldsymbol{\rho}_0$, and $\Delta t_f \equiv t_f$), we have a simple two-point boundary-value problem without an optimization of the velocity change direction. The boundary-value problem is then the following: Given the initial relative position $\boldsymbol{\rho}_0$ and velocity $\boldsymbol{V}_0$, the desired relative position $\boldsymbol{\rho}(t_f) = \boldsymbol{0}$ and the specified impulsive velocity change ΔV, determine the orbit that satisfies these conditions. In a sense, it is a modified Lambert's problem in which the specified transfer time t_f is replaced by a specified impulsive velocity change ΔV. In this case, Eq. (28) is greatly simplified. The distinction between the result presented result here and the approach in Ref. 10 is an explicit solution of the interception problem with specified impulsive velocity change ΔV.

The negative roots represent mathematically possible solutions for homogeneous central gravity field because they give solutions with $\pi/2 < \omega\Delta t_f < \pi$.

The expression for the thrust direction can be written as

$$\boldsymbol{e} = -\frac{\boldsymbol{V}_{kg}\tan\omega\Delta t_f + \boldsymbol{\rho}_{kg}\omega}{W_{vk}\tan\omega\Delta t_f + W_{\rho k}} \tag{29}$$

Second-Order Method for Interception During Free-Flight Phase

For second-order method based on Eqs. (6) and (7), we will be use a constant unit vector of the thrust direction as a near-optimal control law. The solutions of these equations for the constant thrust direction are given in Appendices B and C in the following form:

$$\boldsymbol{\rho}(t) = [\boldsymbol{V}_0/\omega + \boldsymbol{Q}_v(t)]\sin\omega t + [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)]\cos\omega t + \{[W_\rho(t) + S_\rho(t)]/\omega\}\boldsymbol{e} \tag{30}$$

$$\boldsymbol{V}(t) = [\boldsymbol{V}_0 + \boldsymbol{Q}_v(t)\omega]\cos\omega t - [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)]\omega\sin\omega t + [W_v(t) + S_v(t)]\boldsymbol{e} \tag{31}$$

where $\boldsymbol{Q}_v(t)$, $\boldsymbol{Q}_\rho(t)$, $S_\rho(t)$, and $S_v(t)$ are perturbed terms.

The second-order method for free-flight phase interception consists of an expression of the perturbed terms by approximate values, which are calculated along the first-order trajectory:

$$\boldsymbol{\rho}_{kg}^* = \left[\boldsymbol{V}_0/\omega + \boldsymbol{Q}_{va}^*(t_k)\right]\sin\omega t_k + \left[\boldsymbol{\rho}_0 + \boldsymbol{Q}_{\rho a}^*(t_k)\right]\cos\omega t_k + \boldsymbol{Q}_\rho^*\left(\Delta t_f^1\right) \tag{32}$$

$$\boldsymbol{V}_{kg}^* = \left[\boldsymbol{V}_0 + \boldsymbol{Q}_{va}^*(t_k)\omega\right]\cos\omega t_k - \left[\boldsymbol{\rho}_0 + \boldsymbol{Q}_{\rho a}^*(t_k)\right]\omega\sin\omega t_k + \boldsymbol{Q}_v^*\left(\Delta t_f^1\right) \tag{33}$$

where Δt_f^1 is the first-order solution and satisfies the quadratic equation (28) for $\boldsymbol{\rho}_{kg}^*$ and $\boldsymbol{V}_{kg}^*$ and $W_{\rho k}^* = W_{\rho k} + S_\rho(t_k)$ and $W_{vk}^* = W_{vk} + S_v(t_k)$. The thrust direction is computed from Eq. (29).

Numerical experiments show that the second-order method gives reasonably accurate results for the transfer angle less than $\pi/2$. Numerical examples are given in the last section.

Qualitative Analysis of Interception Trajectories

Properties of Interception Trajectories

It is evident that the motion of the interceptor and target during the free-flight phase occurs in a plane that is formed by vectors $\boldsymbol{\rho}_k$ and $\boldsymbol{V}_k$:

$$\boldsymbol{\rho}(t) = \boldsymbol{\alpha}\sin\omega t + \boldsymbol{\beta}\cos\omega t \tag{34}$$

Without loss of generality, by a rotation of the coordinate axes this equation can be reduced to the following two-dimensional forms:

$$\rho_1^* = \alpha_1^*\sin\omega t + \beta_1^*\cos\omega t \tag{35}$$

$$\rho_2^* = \alpha_2^*\sin\omega t + \beta_2^*\cos\omega t \tag{36}$$

After eliminating of the trigonometrical terms, we have a positive definite quadratic form

$$\left(\alpha_2^*\rho_1^* - \alpha_1^*\rho_2^*\right)^2 + \left(-\beta_2^*\rho_1^* + \rho_2^*\beta_1^*\right)^2 = \left(\beta_1^*\alpha_2^* - \beta_2^*\alpha_1^*\right)^2 \tag{37}$$

If $(\beta_1^*\alpha_2^* - \beta_2^*\alpha_1^* = 0)$, then equations

$$\alpha_2^*\rho_1^* - \alpha_1^*\rho_2^* = 0 \tag{38}$$

$$\beta_2^*\rho_1^* - \beta_1^*\rho_2^* = 0 \tag{39}$$

describe a straight line. Thus, the vector $\boldsymbol{\rho}(t)$ or the line of sight remains constant in an inertial frame. It is a well-known guidance law for collision course that is a limiting case of proportional navigation and called the constant bearing navigation.[22]

Phase Portrait and Energy Integral for Relative Motion

Squaring of Eq. (9a), differentiating thrice, and eliminating the trigonometric terms gives

$$\dddot{\rho}\rho + 3\ddot{\rho}\dot{\rho} + 4\omega^2\dot{\rho}\rho = 0 \tag{40}$$

Suppose that $y(\rho) = \dot{\rho}^2(\rho)$. After substitution, we have a second-order differential equation

$$\rho\frac{\mathrm{d}^2 y}{\mathrm{d}\rho^2} + 3\frac{\mathrm{d}y}{\mathrm{d}\rho} + 8\omega^2\rho = 0 \tag{41}$$

which may be integrated to give[23]

$$\dot{\rho}^2 = C_1 + C_2\rho^{-2} - \omega^2\rho^2 \tag{42a}$$

where

$$C_1 = V_k^2 + \rho_k^2\omega^2 \tag{42b}$$

$$C_2 = -\rho_k^2 V_k^2 \sin^2\theta_k \tag{42c}$$

In general case, this solution describes a closed curve that is symmetrical with respect to the ρ axis (Fig. 2a). For the extreme values of ρ, that is, the points with $\dot{\rho} = 0$, we have a biquadratic equation

$$\omega^2\rho^4 - \left(V_k^2 + \omega^2\rho_k^2\right)\rho^2 + \rho_k^2 V_k^2 \sin^2\theta_k = 0 \tag{43}$$

$$\rho_{1,2}^2 = \left[\left(V_k^2 + \rho_2^2\omega^2\right) \pm \sqrt{\left(V_k^2 + \rho_k^2\omega^2\right)^2 - 4\omega^2\rho_k^2 V_k^2 \sin^2\theta_k} \Big/ 2\omega^2\right] \tag{44}$$

Note that all of the roots are nonnegative because for the extreme value of $\sin^2\theta_k = 1$ the discriminant is a square, $D = (V_k^2 - \rho_k^2\omega^2)^2 > 0$. For a collision course, that is, $\theta_k = \pi$ (Fig. 2b), Eq. (44) can be written as

$$\rho_{\max} = \sqrt{V_k^2 / \omega^2 + \rho_k^2}, \qquad \rho_{\min} = 0 \tag{45}$$

and when $\theta_k = \pi/2$, it is

$$\rho_1 = V_k/\omega, \qquad \rho_2 = \rho_k \tag{46}$$

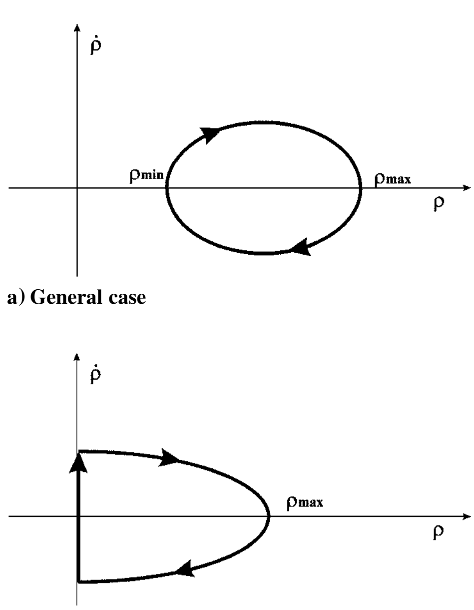

Fig. 2 Phase portrait.

For the sum squares of $\boldsymbol{\rho}(t)$ and $\boldsymbol{V}(t)$, we obtain

$$V^2(t) + \rho^2(t)\omega^2 = V_k^2 + \rho_k^2\omega^2 = q = \text{const} \tag{47}$$

For relative motion, we can identity $V^2(t)$ as the kinetic energy per unit mass and $\rho^2(t)\omega^2$ as the potential energy. Thus, as in inverse-square orbital motion, the quantity q can be called the total energy constant of relative motion. It is evident that the final collision velocity between the interceptor and target is

$$V_f = \sqrt{q} = \sqrt{V_k^2 + \rho_k^2\omega^2} \tag{48}$$

Geometric Interpretation of Solutions

As already mentioned, the collision course required collinear vectors of $\boldsymbol{V}_k$ and $\boldsymbol{\rho}_k$. We consider a case with $\theta_k > \pi/2$ and $|\boldsymbol{V}_k| > W_{vk}$. Figure 3 shows the vectors $\boldsymbol{\rho}_{kg}$ and $\boldsymbol{V}_{kg}$ and two circles centered at A and B having radii $W_{\rho k}/\omega$ and W_{vk}, respectively. By rotating, we find a position of a line passing through point O that must satisfy the conditions of BC∥AE and BD∥AF. This means that there are two solutions for interception trajectories with transfer times $\omega\Delta t_{f1,2} < \pi/2$ and that the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the required thrust directions. It is evident that there is a value of θ_k with a boundary solution (Fig. 4). Finally, for $\theta_k < \pi/2$, it can be shown (Fig. 5) that there are two solutions with $\pi/2 < \omega\Delta t_f < \pi$. If $W_{vk} > |\boldsymbol{V}_k|$, there are always two solutions for transfer times $\omega\Delta t_{f1} < \pi/2$ and $\pi/2 < \omega\Delta t_f < \pi$ (Fig. 6).

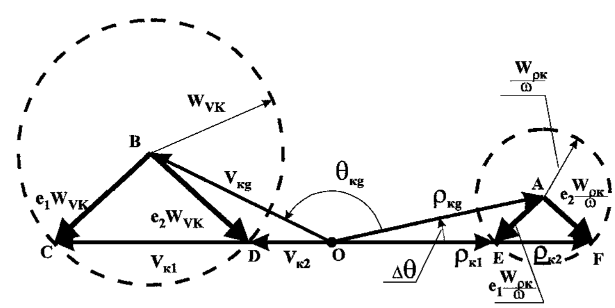

Fig. 3 Geometric interpretation of solutions for $\theta_{kg} > \pi/2$ and $V_{kg} > W_{vk}$.

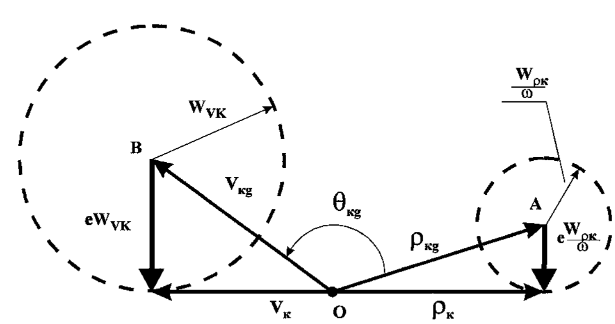

Fig. 4 Boundary solution for $\theta_{kg} > \pi/2$ and $V_{kg} > W_{vk}$.

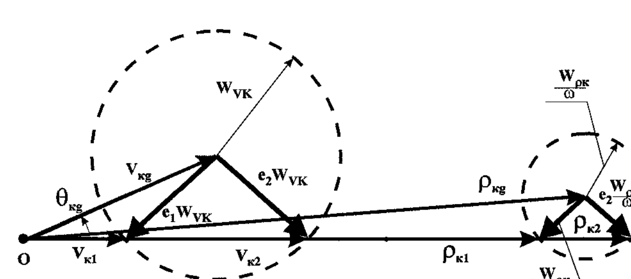

Fig. 5 Solutions for $\theta_{kg} < \pi/2$.

Table 1 Solution types

θ_{kg}	Relation for V_{kg} and W_{vk}	Existence condition	Number of solutions: $\omega\Delta t_f < \pi/2$	Number of solutions: $\pi/2 < \omega\Delta t_f < \pi$
$\theta_{kg} > \pi/2$	$V_{kg} > W_{vk}$	$V_{kg}\sin(\theta + \Delta\theta_{\max}) \le W_{vk}$	2	——
$\theta_{kg} > \pi/2$	$V_{kg} > W_{vk}$	$V_{kg}\sin(\theta + \Delta\theta_{\max}) \le W_{vk}$	——	2
$0 \le \theta_{kg} \le \pi$	$V_{kg} < W_{vk}$	Always	1	1
Arbitrary	$V_{kg} = W_{vk}$	Always	1	$\omega\Delta t_f = \pi/2$

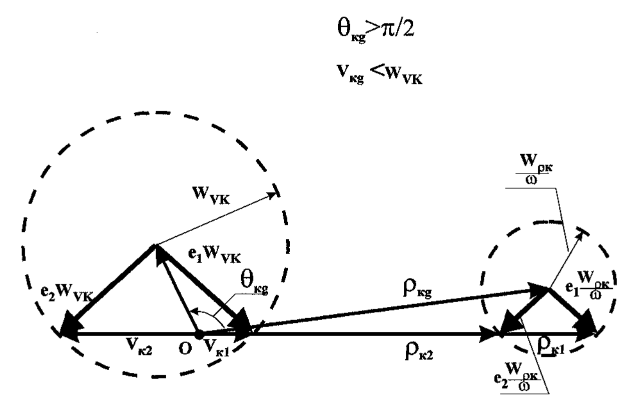

Fig. 6 Solutions for $W_{vk} > V_{kg}$.

From the earlier geometric consideration, it is easily shown that the existence condition for solutions is

$$V_{kg}\sin(\theta_{kg} + \Delta\theta_{\max}) \le W_{vk} \tag{49}$$

where

$$\Delta\theta_{\max} = \sin^{-1}(W_{\rho k}/\rho_{kg}\omega) \tag{50}$$

This relation can be derived analytically. For impulsive velocity change it is

$$V_0 \sin\theta_0 \le \Delta V \tag{51}$$

Clearly the existence of a solution means the energetic attainability of the target. Table 1 presents the conditions for distinct solution types.

Reachable Domains for Fixed-Fuel Interception

We defined a reachable domain as a boundary set of the initial positions of an interceptor and a target for which the target is energetic attainable at a time to the intercept free flying about a spherical Earth and using a fixed-fuel potential represented by a specified W_{vk} or impulsive velocity change ΔV. Based on the preceding analysis, the definition of the reachable domains is required computation of the areas in a space with the following boundaries: $W_{vk} = V_{kg}\sin\theta_{kg}$, $W_{vk} = V_{kg}$, and $\theta_{kg} = \pi/2$. If initial relative state between the interceptor and target can be defined in a two-dimensional space, then contour diagrams are very useful. Corresponding examples are given in the next section.

Application Examples

Near-Earth Asteroid Interception

The problem of interception of asteroids on collision courses with the Earth has become of increasing importance in recent years.[24–26] A traditional method for analysis of interplanetary trajectories is the patched conic method.[6,14] In this method, if the spacecraft is sufficiently close to the Earth, then in a region called the sphere of influence of the Earth can be analyzed, as the two-body Earth–spacecraft problem. If the spacecraft is not inside the sphere of influence of the Earth, it is considered to be in a two-body orbit about the sun. The Earth's sphere of influence radius is 6×10^{-3} astronomical unit (AU). Suppose that $\boldsymbol{V}_E$ is the Earth velocity vector relative to the sun; thus, the spacecraft heliocentric velocity $\boldsymbol{V}_{SV}$ is a sum $\boldsymbol{V}_{SV} = \boldsymbol{V}_E + \boldsymbol{V}_{SE}$, where $\boldsymbol{V}_{SE}$ is the velocity of the spacecraft relative to the Earth, which often called the hyperbolic excess velocity vector. It is known[6,14] that the required excess velocity for the Earth escape hyperbola is precisely the same as the impulsive velocity change ΔV for a massless planet. For known $\boldsymbol{V}_{SE}$, the actual required ΔV can be determined. We assume that the velocity $\boldsymbol{V}_{SE}$ has an approximate value of ΔV. It is equivalent to an interceptor guided from a massless planet. Similar to near-Earth trajectories, suppose that the motion of the interceptor and an asteroid occurs in a thin spherical shell with heliocentric radius near to the mean heliocentric radius of the Earth's orbit.

Table 2 Numerical example of asteroid interception

Solution type	Parameters: t_f, days	Parameters: $\boldsymbol{e}$	Errors: $\Delta\rho_{\min}$	Errors: δt_f	Errors: Δe, deg
First order	60.15	$(-0.1106, -0.9938)^T$	0.081	0.015	12.6
Second order	60.05	$(-0.2966, -0.9550)^T$	0.010	0.013	1.73
Exact solution	59.25	$(-0.3254, -0.9456)^T$	——	——	——

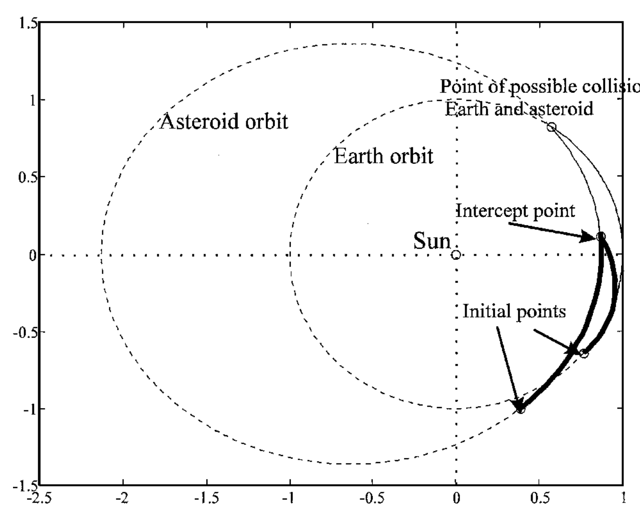

Fig. 7 Example of asteroid interception trajectory.

Consider an Earth-crossing asteroid, moving in the ecliptic plane, whose orbital elements are given by semimajor axis of 1.5 AU and eccentricity of 0.42. Suppose that initial positions of the asteroid and an interceptor with $\Delta V = 8$ km/s in a two-dimensional heliocentric coordinate frame are $\boldsymbol{r}_{T0} = (0.3901, -0.9980)^T$ AU and $\boldsymbol{r}_{I0} = (0.7633, -0.6405)^T$ AU. The time before collision of the Earth and asteroid is $t_c = 97$ days. One of the first-order solutions of Eq. (28) is a negative root $\omega t_{f1} = -77.36$ deg, which corresponds to interception time $t_{f1} = 102.9$ days (for $r_m = 1.0$ AU). It is evident that this solution with $t_{f1} > t_c$ is not of a practical significance. Trajectory in the ecliptic plane corresponding to the second solution with positive root ωt_{f2} is shown in Fig. 7. Parameters of the first- and second-order solutions and their errors relative an exact solution for inverse-square gravity field are given in Table 2. The minimal range error $\delta\rho_{\min} = \rho_{\min}/\rho_0$ and time error $\delta t_f = (t_f - t'_f)/t'_f$ provided by the substitution of approximate solutions, that is, the corresponding vector $\boldsymbol{e}$, in the equations of the asteroid and interceptor for inverse-square gravity field, where $\rho_{\min}$ is the minimal range between them and t'_f is corresponding time. The thrust direction error Δe is an angle between the vectors $\boldsymbol{e}$ for approximate solutions and the exact solution, which here is computed as a two-point boundary value problem for inverse-square gravity field, that is, the problem is to find $\boldsymbol{e}$ and t_f for specified $\boldsymbol{\rho}_0$, $\boldsymbol{V}_0$, and $\boldsymbol{\rho}_f \equiv \boldsymbol{0}$.

Reachable domain and transfer times vs start time before collision of the Earth and asteroid, t_c, for $\Delta V = 2$–22 km/s are presented in Fig. 8. It shown that there exists an area for small ΔV in which there are two solutions with $\omega t_f < \pi/2$. However, only solutions with $t_f < t_c$ are of practical significance.

Space Interception

As second example, consider a fictitious space-based interceptor and a target both moving in circular low Earth orbits of radii $r_{I0} = 6678$ km and $r_{T0} = 6728$ km, respectively. Suppose that the interceptor parameters are given by the $a = 0.006$ km/s^2, $t_k = 300$ s, and $\dot{m} = 0.002667$ 1/s. Substituting these parameters into Eqs. (A7) yields $W_{vk} = 3.576$ km/s and $W_{\rho k} = 0.458$ km/s (for $r_m = 6728$ km). We define an Earth-centered inertial coordinate frame as shown in Fig. 9. The X axis is placed along the line of intersection of the two orbital planes, and the Y axis lies at the orbital plane of the interceptor. The relative state vector between the interceptor and target is specified by three parameters: the inclination angle of the interceptor orbital plane from that of the target Δi and the angles to the interceptor and target from the node of intersection of their orbital planes φ_I and φ_T, respectively. Therefore,

$$\boldsymbol{\rho}_0 = r_T \begin{bmatrix} \cos\varphi_T \\ \sin\varphi_T \cos\Delta i \\ \sin\varphi_T \sin\Delta i \end{bmatrix} - r_I \begin{bmatrix} \cos\varphi_I \\ \sin\varphi_I \\ 0 \end{bmatrix}$$

$$\boldsymbol{V}_0 = V_T \begin{bmatrix} -\sin\varphi_T \\ \cos\varphi_T \cos\Delta i \\ \cos\varphi_T \sin\Delta i \end{bmatrix} - V_I \begin{bmatrix} -\sin\varphi_I \\ \cos\varphi_I \\ 0 \end{bmatrix} \tag{52}$$

where V_T and V_I orbital velocities.

Consider the interception problem for a particular case of $\varphi_T = -110$ deg, $\varphi_I = -30$ deg, and $\Delta i = 150$ deg. In this case, we have from Eq. (28) two positive roots for the first- and also for the second-order solutions. Parameters of trajectories and their errors are presented in Table 3.

An example of reachable domains for arbitrary φ_I and φ_T and $\Delta i = 150$ deg are shown in Fig. 10. Because $W_{\rho k}/\omega = 400$ km $> r_T - r_I$ for all values $\Delta\varphi$ and Δi, there are small areas of boost-phase interception near nodes of the orbit planes intersection. The solutions for these areas can be computed using Eq. (18). A bound of this area is a bound of area of interception during free-flight phase with two solutions in which $\omega\Delta t_f \le \pi/2$. Another bound of the last area is a bound of area with two solutions in which $\omega\Delta t_{f1} \le \pi/2$ and $\pi/2 < \omega\Delta t_{f2} < \pi$. In closing, we have an area with the solutions of $\omega\Delta t_f > \pi/2$. Examples of reachable domains for some inclinations Δi are shown in Fig. 11. Boundary values of $\Delta i = 0$ and π conform to coplanar interception. With inclination increasing, the reachable domains increased. For $\Delta i = 0$, the reachable domain is a small area corresponding to relative positions of the interceptor and target at a short distance (Fig. 11a). For $\Delta i = \pi$ there is an opposite case,

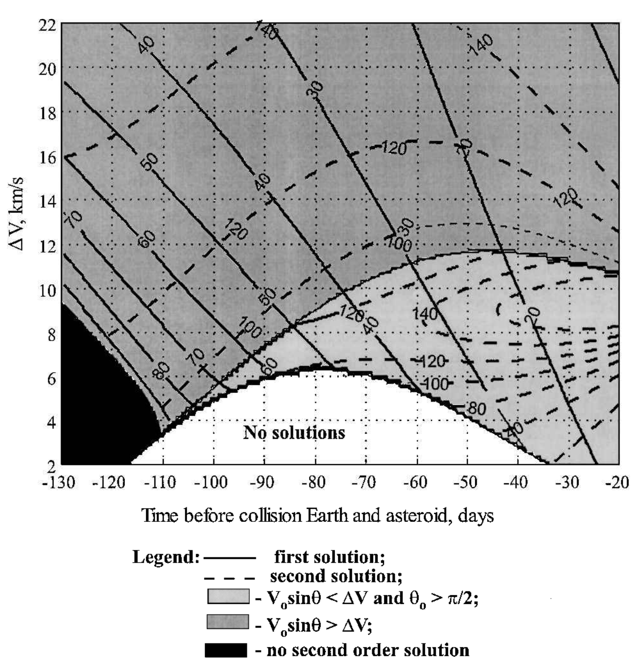

Fig. 8 Reachable domain and contour plot of transfer times (days) for trajectories of asteroid interception.

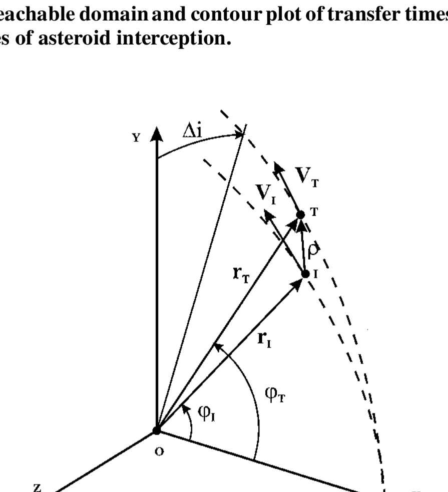

Fig. 9 Coordinate frame for space interception.

Table 3 Example of space interception

Parameter and error	First solution	Second solution
	First order	
t_f, s	928.0	589.9
$\boldsymbol{e}$	$(0.5510, -0.8068, -0.2134)^T$	$(-0.7514, 0.5877, 0.2961)^T$
$\delta\rho_{\min}$	0.152	0.011
δt_f	0.045	0.066
	Second order	
t_f, s	902.15	560.1
$\boldsymbol{e}$	$(0.0894, -0.9637, -0.2517)^T$	$(-0.7273, 0.6155, 0.3038)^T$
$\delta\rho_{\min}$	0.032	0.0023
Δt_f	0.061	0.012

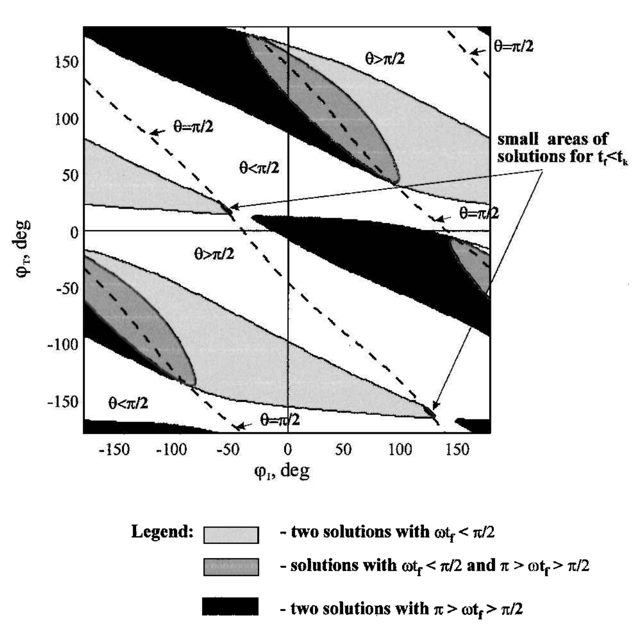

Fig. 10 Reachable domain of space interception for Δ i = 150 deg.

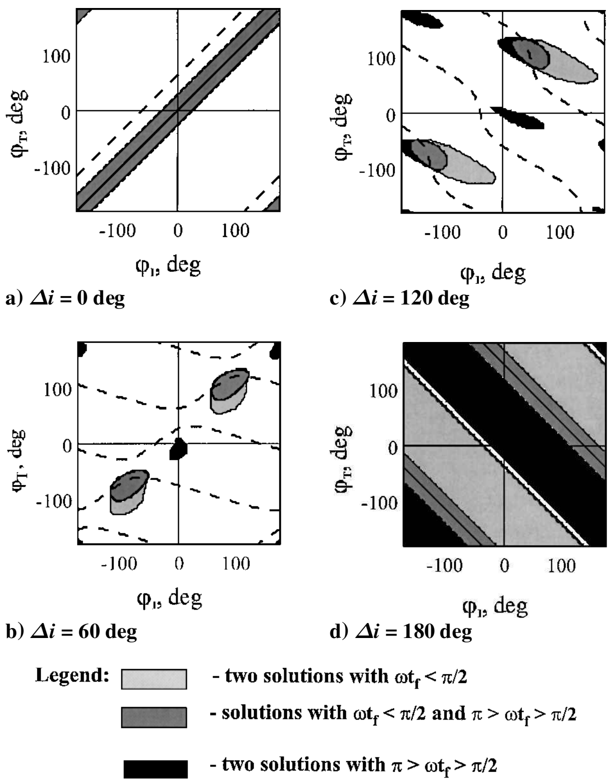

Fig. 11 Reachable domains for several Δ *i*.

that is, it is a small area in which the solutions are not available (Fig. 11d).

Conclusions

A simple model for relative motion of an interceptor and target moving in a thin spherical shell including an attracting spherical homogeneous body or point mass has been used to obtain closed-form solutions of a fixed-fuel interception with a free transfer time. The major results of the paper may be summarized as follows: 1) A quadratic equation for the transfer time of interception has been derived. 2) The first- and second-order solutions have been obtained. 3) A general formulation of the solutions for an intercept with constant thrust has been established; the solution with impulsive thrust approximation is a particular case of this general case. 4) The methods give reasonably accurate results for the transfer angle less than $\pi/2$. 5) Based on descriptive existence conditions, a computational method for the reachable domain is developed.

The rationale of the method is to obtain a better understanding of the nature of space interception. This knowledge should be useful for computing of reachable domains. Furthermore, approximate methods often serve a useful role as a starting of reference solutions for higher accuracy approaches.

Appendix A: Analytic Solution for the Boost-Phase with Constant Thrust

Equations (8) may be written as a second-order differential equation:

$$\ddot{\boldsymbol{\rho}} + \omega^2 \boldsymbol{\rho} = -[a\boldsymbol{e}/(1 - \dot{m}t)] \tag{A1}$$

By the use of the method of undetermined coefficients, a particular solution of the homogeneous equation (A1) may be written as

$$\boldsymbol{\rho}(t) = \boldsymbol{\alpha} \sin \omega t + \boldsymbol{\beta} \cos \omega t + (a\boldsymbol{e}/\omega\dot{m}) f_\rho(t)$$
$$\boldsymbol{V}(t) = \boldsymbol{\alpha}\omega \cos \omega t - \boldsymbol{\beta}\omega \sin \omega t + (a\boldsymbol{e}/\dot{m}) f_v(t) \tag{A2}$$

where $f_\rho(t)$ and $f_v(t)$ are dimensionless functions:

$$f_\rho(t) = Ci(u) \sin u - Si(u) \cos u$$
$$f_v(t) = Ci(u) \cos u + Si(u) \sin u \tag{A3}$$

where $Ci(u)$ and $Si(u)$ are cosine and sine integral functions,[27] respectively, with argument

$$u = \omega t - \omega/\dot{m} \tag{A4}$$

Finally, we have

$$\boldsymbol{\rho}(t) = (\boldsymbol{V}_0/\omega) \sin \omega t + \boldsymbol{\rho}_0 \cos \omega t + [W_\rho(t, a, \dot{m})/\omega]\boldsymbol{e}$$
$$\boldsymbol{V}(t) = \boldsymbol{V}_0 \cos \omega t - \boldsymbol{\rho}_0 \omega \sin \omega t + W_v(t, a, \dot{m})\boldsymbol{e} \tag{A5}$$

where $W_\rho(t)$ and $W_v(t)$ are

$$W_\rho(t, a, \dot{m}) = (a/\dot{m})[f_\rho(t) - f_\rho(0) \cos \omega t - f_v(0) \sin \omega t]$$
$$W_v(t, a, \dot{m}) = (a/\dot{m})[f_v(t) + f_\rho(0) \sin \omega t - f_v(0) \cos \omega t] \tag{A6}$$

Expressions (A6) are the scaled distance and velocity change for the homogeneous central gravity field. The last expression in Eq. (A6), W_v, represents the change in the velocity magnitude due to the thrust force.

Appendix B: Second-Order Solution for Free-Flight Phase

From the first-order solution for the interceptor and target, we have

$$\boldsymbol{r}_T(t) = (\boldsymbol{V}_{T0}/\omega) \sin \omega t + \boldsymbol{r}_{T0} \cos \omega t \tag{B1}$$

$$\boldsymbol{r}_I(t) = (\boldsymbol{V}_{I0}/\omega) \sin \omega t + \boldsymbol{r}_{I0} \cos \omega t \tag{B2}$$

For $\delta r_T(t) = r_T(t) - r_m$ and $\delta r_I(t) = r_I(t) - r_m$, we make the following quadratic approximation:

$$\delta r = a_0 + a_1 t + a_2 t^2 \tag{B3}$$

$$a_0 = r_0 - r_m \tag{B4}$$

$$a_1 = \boldsymbol{V}_0^T \boldsymbol{r}_0 \big/ |r_0| = V_{r0} \tag{B5}$$

$$a_2 = (r_f - r_0 - V_{r0} t_f) \big/ t_f^2 \tag{B6}$$

Using the method of undetermined coefficients, we have the following second-order solution:

$$\boldsymbol{\rho}(t) = [\boldsymbol{V}_0/\omega + \boldsymbol{Q}_v(t)] \sin \omega t + [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)] \cos \omega t_k \tag{B7}$$

$$\boldsymbol{V}(t) = [\boldsymbol{V}_0 + \boldsymbol{Q}_v(t)\omega] \cos \omega t - [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)]\omega \sin \omega t_k \tag{B8}$$

where

$$\boldsymbol{Q}_\rho = (3\omega/r_m)[\boldsymbol{d}_\beta(0) - \boldsymbol{d}_\beta(t)] \tag{B9}$$

$$\boldsymbol{Q}_v = (3\omega/r_m)[\boldsymbol{d}_\alpha(t) - \boldsymbol{d}_\alpha(0)] \tag{B10}$$

$$\boldsymbol{d}_\alpha(t) = \sum_{i=0}^{i=2} \left(\boldsymbol{a}_i^v I_{SCi} + \boldsymbol{a}_i^r I_{Ci}\right) \tag{B11}$$

$$\boldsymbol{d}_\beta(t) = \sum_{i=0}^{i=2} \left(\boldsymbol{a}_i^v I_{Si} + \boldsymbol{a}_i^r I_{SCi}\right) \tag{B12}$$

$$\boldsymbol{a}_i^v = (a_{iT} \boldsymbol{V}_{I0} - a_{iI} \boldsymbol{V}_{T0})/\omega \tag{B13}$$

$$\boldsymbol{a}_i^r = a_{iT} \boldsymbol{r}_{T0} - a_{iI} \boldsymbol{r}_{I0} \tag{B14}$$

$$I_{S0} = \int \sin^2 \omega t \, \mathrm{d}t = \frac{1}{\omega}\left(\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right) \tag{B15}$$

$$I_{C0} = \int \cos^2\omega t \,\mathrm{d}t = \frac{1}{\omega}\left(\frac{\omega t}{2} + \frac{\sin 2\omega t}{4}\right) \tag{B16}$$

$$I_{SC0} = \frac{1}{2}\int \sin 2\omega t \,\mathrm{d}t = -\frac{1}{4\omega}\cos 2\omega t \tag{B17}$$

$$I_{SI} = \int t\sin^2\omega t \,\mathrm{d}t = \frac{1}{\omega^2}\left(\frac{\omega^2 t^2}{4} - \frac{\omega t \sin 2\omega t}{4} - \frac{\cos 2\omega t}{8}\right) \tag{B18}$$

$$I_{C1} = \int t\cos^2\omega t \,\mathrm{d}t = \frac{1}{\omega^2}\left(\frac{\omega^2 t^2}{4} + \frac{\omega t \sin 2\omega t}{4} + \frac{\cos 2\omega t}{8}\right) \tag{B19}$$

$$I_{SC1} = \frac{1}{2}\int t\cos 2\omega t \,\mathrm{d}t = \frac{1}{8\omega^2}(\sin 2\omega t - 2\omega t\cos 2\omega t) \tag{B20}$$

$$I_{S2} = \int t^2\sin^2\omega t \,\mathrm{d}t$$

$$= \frac{1}{\omega^3}\left[\frac{\omega^3 t^3}{6} - \left(\frac{\omega^2 t^2}{4} - \frac{1}{8}\right)\sin 2\omega t - \frac{\omega t\cos 2\omega t}{4}\right] \tag{B21}$$

$$I_{C2} = \int t^2\cos^2\omega t \,\mathrm{d}t$$

$$= \frac{1}{\omega^3}\left[\frac{\omega^3 t^3}{6} + \left(\frac{\omega^2 t^2}{4} - \frac{1}{8}\right)\sin 2\omega t + \frac{\omega t\cos 2\omega t}{4}\right] \tag{B22}$$

$$I_{SC2} = \frac{1}{2}\int t^2\sin 2\omega t \,\mathrm{d}t$$

$$= \frac{1}{16\omega^3}[4\omega t\sin 2\omega t - (4\omega^2 t^2 - 2)\cos 2\omega t] \tag{B23}$$

Appendix C: Second-Order Solution for Boost-Phase

In a similar manner, we can use the corresponding expression for the boost phase of the interceptor:

$$\boldsymbol{r}_I(t) = (\boldsymbol{V}_{I0}/\omega)\sin\omega t + \boldsymbol{r}_{I0}\cos\omega t + [W_\rho(t)/\omega]\boldsymbol{e} \tag{C1}$$

and the following quadratic approximation of $W_\rho(t)$:

$$W_\rho(t) \approx b_1 t + b_2 t^2 \qquad (b_0 \equiv 0) \tag{C2}$$

Similar to Appendices A and B, the second-order solution for the boost phase can be written as

$$\boldsymbol{\rho}(t) = [\boldsymbol{V}_0/\omega + \boldsymbol{Q}_v(t)]\sin\omega t + [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)]\cos\omega t$$

$$+ \{[W_\rho(t) + S_\rho(t)]/\omega\}\boldsymbol{e} \tag{C3}$$

$$\boldsymbol{V}(t) = [\boldsymbol{V}_0 + \boldsymbol{Q}_v(t)\omega]\cos\omega t - [\boldsymbol{\rho}_0 + \boldsymbol{Q}_\rho(t)]\omega\sin\omega t$$

$$+ [W_v(t) + S_v(t)]\boldsymbol{e} \tag{C4}$$

where $\boldsymbol{Q}_v(t)$ and $\boldsymbol{Q}_\rho(t)$ are the perturbed terms of Eqs. (B9) and (B10):

$$S_\rho(t) = (3/r_m)\{[c_\alpha(0) - c_\alpha(t)]\sin\omega t + [c_\beta(t) - c_\beta(0)]\cos\omega t\} \tag{C5}$$

$$S_v(t) = (3/r_m)\{[c_\alpha(0) - c_\alpha(t)]\cos\omega t + [c_\beta(t) - c_\beta(0)]\sin\omega t\} \tag{C6}$$

$$C_\alpha(t) = S_1(\cos\omega t + \omega t\sin\omega t) + S_2[2\omega t\cos\omega t$$

$$+ (\omega^2 t^2 - 2)\sin\omega t] + S_3[(3\omega^2 t^2 - 6)\cos\omega t$$

$$+ (\omega^3 t^3 - 6\omega t)\sin\omega t] + S_4[(4\omega^3 t^3 - 24\omega t)\cos\omega t$$

$$+ (\omega^4 t^4 - 12\omega^2 t^2 + 24)\sin\omega t] \tag{C7}$$

$$C_\beta(t) = S_1(\sin\omega t - \omega t\cos\omega t) + S_2[2\omega t\sin\omega t$$

$$- (\omega^2 t^2 - 2)\cos\omega t] + S_3[(3\omega^2 t^2 - 6)\sin\omega t$$

$$- (\omega^3 t^3 - 6\omega t)\sin\omega t] + S_4[(4\omega^3 t^3 - 24\omega t)\sin\omega t$$

$$+ (\omega^4 t^4 - 2\omega^2 t^2 + 24)\cos\omega t] \tag{C8}$$

$$S_1 = a_{I0}b \tag{C9}$$

$$S_2 = (a_{I1}b_1 + a_{I0}b_2)/\omega \tag{C10}$$

$$S_3 = (a_{I2}b_1 + a_{I0}b_2)\big/\omega^2 \tag{C11}$$

$$S_4 = a_{I2}b_2\big/\omega^3 \tag{C12}$$

References

[1]Doll, J. R., and Gobetz, F. W., "Survey of Impulsive Trajectories," *AIAA Journal*, Vol. 7, No. 5 , 1969, pp. 801–834.

[2]Bryson, A. E., Jr., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, New York, 1975, Chaps. 2, 3.

[3]Ilin, V. A., and Kuzmak, G. E., *Optimal Transfer Orbits of Spacecraft with High-Thrust Engines*, Izdatel'stvo Nauka, Moscow, 1976, Chaps. 8–9 (in Russian).

[4]Salama, A. I. A., and Hamza, M. H., "Minimum Time Three-Dimensional Interception," *Acta Astronautica*, Vol. 5, No. 4, 1978, pp. 515–522.

[5]Agaponov, S. V., "Globally Optimal Coplanar Time-Free Hard Rendezvous," *Leningradskii Universitet Vestnik (Serie Matematika, Mekhanika, Astronomiia)*, No. 2, 1985, pp. 83–85 (in Russian).

[6]Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, New York, 1987, Chaps. 6, 11.

[7]Prussing, J. E., Wellnitz, L. J., and Heckathorn, W. G., "Optimal Impulsive Time-Fixed Direct-Ascent Interception," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 4, 1989, pp. 487–494.

[8]Parks, A. D., Lawton, J. A., and Bell, D. B., "Simple Hybrid Search Technique for Finding Conic Solutions to the Intercept Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 5, 1989, pp. 764–766.

[9]Lembeck, C. A., and Prussing, J. E., "Optimal Impulsive Intercept with Low-Thrust Rendezvous Return," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 3, 1993, pp. 426–433.

[10]Cochran, J. E., Jr., and Haynes, D. A., "Constrained Initial Guidance Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 193–197.

[11]Vinh, N. X., Lu, P., Howe, R. M., and Gilbert, E. G., "Optimal Interception with Time Constraint," *Journal of Optimization Theory and Applications*, Vol. 66, No. 3, 1990, pp. 361–390.

[12]Vinh, N. X., Gilbert, E. G., Howe, R. M., Sheu, D., and Lu, P., "Reachable Domain for Interception at Hyperbolic Speeds," *AIAA/AAS Astrodynamics Conference: Technical Papers*, Pt. 2, AIAA, Washington, DC, 1990, pp. 907–915.

[13]Emel'yanov, D. D., "Concerning a Particular Problem of Interception," *Avtomatika i Telemekhanika*, No. 6, 1993, pp. 85–94 (in Russian).

[14]Prussing, J. E., and Conway, W., *Orbital Mechanics*, Oxford Univ. Press, Oxford, 1993, Chaps. 4, 6.

[15]Berdyshev, Y. I., "On Problems of One Impulse Transfer and Constructing of Safety Domain in Newtonian Field," *Kosmicheskie Issledovaniia*, Vol. 31, No. 6, 1993, pp. 3–10 (in Russian); *Cosmic Research*, Vol. 31, No. 6, 1993, pp. 3–9 (English translation).

[16]Clohessy, W. H., and Wiltshire, R. S., "Terminal Guidance Systems for Satellite Rendezvous," *Journal of the Aerospace Sciences*, Vol. 27, Sept. 1960, pp. 653–658, 674.

[17]Davidson, B. K., Isaev, V. K., and Kuzmak, G. E., "Optimality Conditions for the Motion of a Point of Variable Mass in a Homogeneous Central Field," *Soviet Physics—Doklady*, Vol. 8, No. 9, 1963, pp. 255–259.

[18]Kuzmak, G. E., and Braude, A. Z., "An Investigation of Approximate Optimal Control of Motion of a Material Point in a Thin Spherical Layer of Central Gravitational Field," *Kosmicheskie Issledovaniia*, Vol. 9, March–April 1971, pp. 195–210 (in Russian); *Cosmic Research*, Vol. 9, Oct. 1971, pp. 182–195 (English translation).

[19]Jezewski, D. J., "An Optimal, Analytic Solution to the Linear-Gravity, Constant-Thrust Trajectory Problem," *Journal of Spacecraft and Rockets*, Vol. 8, No. 7, 1971, pp. 793–796.

[20]Kirpichnikov, S. N., "Attainability Region for Single-Impulse Transfer from a Keplerian Orbit," *Leningradskii Universitet Vestnik (Serie Matematika, Mekhanika, Astronomiia)*, No. 4, 1990, pp. 42–46 (in Russian).

[21] Vinh, N. X., Gilbert, E. G., Howe, R. M., Sheu, D., and Lu, P., "Reachable Domain for Interception at Hyperbolic Speeds," *AIAA/AAS Astrodynamics Conference: Technical Papers*, Pt. 2, AIAA, Washington, DC, 1990, pp. 907–915.

[22] Houbolt, J. C., "Problems and Potentialities of Space Rendezvous," *Astronautica Acta*, Vol. 7, No. 5–6, 1961, pp. 406–429.

[23] Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen*, Chelsea, New York, 1959, Chap. 3.

[24] Solem, J. C., "Interception of Comets and Asteroids on Collision Course with Earth," *Journal of Spacecraft and Rockets*, Vol. 30, No. 2, 1993, pp. 222–228.

[25] Park, S.-Y., and Ross, M. I., "Two-Body Optimization for Deflecting Earth-Crossing Asteroids," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 3, 1999, pp. 415–420.

[26] Ivashkin, V. V., and Smirnov, V. V., "An Analysis of Some Methods of Asteroid Hazard Mitigation for the Earth," *Planetary and Space Science*, Vol. 43, No. 6, 1995, pp. 821–825.

[27] Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series, and Products*, 5th ed., Academic Press, London, 1993, Chap. 8.23.

D. B. Spencer
Associate Editor